AIAA JOURNAL
Vol. 40, No. 1, January 2002

Displacement-Based Optimization for Truss Structures Subjected to Static and Dynamic Constraints

Samy Missoum* and Zafer Gürdal†
Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061

The use of displacements as design variables for truss structure optimization is considered as an alternative to the conventional finite-element-analysis-based design approach. A two-level nested optimization has been developed; in an inner level, cross-sectional areas of truss members are designed for a given displacement field, and, at an outer level, optimal displacements corresponding to the minimum weight designs are searched through the use of sequential quadratic programming. The computational efficiency of the method is demonstrated through three examples, and the evolutions of the cross-sectional areas and optimal weight as a function of the displacements are studied. Static, dynamic, and topology problems are considered. It is shown that the method is highly efficient compared to the conventional design approaches. It is also demonstrated that the weight is always continuous throughout the design history, although areas might exhibit large discontinuities.

Introduction

FOR the past 30 years, optimization using sizing variables such as areas or thicknesses has been the most popular form of design optimization problem formulation. The primary goal (objective function) of such optimization problems may be to minimize the weight or to maximize a natural frequency of a structure under static or dynamic constraints. A typical strategy for the solution of the problem has been to use an iterative approach where the optimum is found by solving the structural analysis equations repetitively until the objective function cannot be improved and none of the constraints are violated. Depending on the number of variables, such an iterative technique may require a large number of structural analyses that are generally costly and time consuming. The cost of the structural analysis depends on the type of analysis required to determine the constraints and/or the objective function. For example, nonlinear structural analysis of complex structural shapes using finite element method procedures is a real handicap for the structural optimization process.

For those cases in which the analysis becomes complex and computationally expensive, a different solution scheme was proposed in the literature to improve the efficiency of the process. The approach relies on formulating the problem such that the structural analysis equations are added to the constraints of the original optimization problem as equality constraints. The approach is commonly referred to as the simultaneous analysis and design (SAND) procedure[1,2] and requires only the solution of one nonlinear optimization problem without requiring a complete structural analysis at every stage of the design optimization. The method is particularly well suited for problems that require nonlinear analysis because the additional burden of solving nonlinear equilibrium equations can be handled as part of the nonlinear optimization problem. In the SAND approach, the displacements, as well as the conventional sizing variables, are treated as the unknown variables of the optimization problem.

As an alternative to the classical optimization and SAND schemes, there may be an approach in which only the displacements are used as unknown optimization variables. In this case, the optimization process will search for the optimal structure in the displacement space only and not in the sizing-variable space. However, the goal of the optimization process being the definition of an optimal set of sizing quantities, a two-level optimization procedure has to be built so that the optimal displacements are searched and the corresponding set of sizing variables is found. These two levels will be referred to as inner and outer problems, respectively. Because the search is performed in the displacement space, the method is referred to as the displacement-based optimization. For the set of displacement variables specified at any stage of the optimization process, the sizing variables are to be solved for as an inner problem. That is, the method reverses the traditional way of nesting the solution of the displacements inside the solution of the design variables. Instead, the outer optimization loop is in terms of the displacements while the solution of the design variables is nested. This represents the major difference between the classical sizing and the displacement-based optimization. In classical sizing, at any stage of the optimization, sizing variables are specified, and the analysis provides the solution for the displacements. In the displacement-based approach, the scheme is reversed. The solution of the sizing variables is sought at each stage of the optimization in which the displacement variables are specified.

There are several motivations for developing a displacement-based optimization. The most important motivation is the expected computational efficiency of the method. Indeed, in the traditional finite-element (FE)-based approach, the use of repeated analysis for different values of the sizing variables can be an expensive task. Therefore, the solution of a system in which displacements are specified and the sizing variables are the unknown variables is computationally cheaper than solving a system in which the sizing variables are specified and the displacements are unknown.

Another motivation for the method is related to its use in the topology optimization. The primary objective of topology optimization is to remove parts of a structure so that a certain optimality criterion is satisfied. When using a FE- analysis-based approach, topology problems become quite tedious because the reduction of cross-sectional dimensions often increases the stresses. This forces the stress constraints to become active and start driving the design away from reducing the thicknesses. Therefore, unless the sizing variables are assigned a zero value, they may never approach zero. On the other hand, if the sizing variables are assigned a zero value, the stiffness matrix becomes singular unless the definition of the element is removed from the analysis model.

Because of the difficulty described, Ringertz[3] has worked on a two-stage procedure based on the use of displacements, which is constructed as follows:

1) Find the topology of a truss for a given displacement field to minimize its weight.

Presented as Paper 98-3977 at the 7th Multidisciplinary Analysis and Optimization Conference, St. Louis, MO, 2–4 September 1998; received 29 September 2000; revision received 1 June 2001; accepted for publication 12 June 2001.

*Research Associate, Aerospace and Ocean Engineering.
†Professor, Aerospace and Ocean Engineering, and Engineering Science and Mechanics Departments. Associate Fellow AIAA.

2) For the topology determined in the first step, find the cross-sectional areas that minimize the weight while satisfying the stress and displacement constraints.

The distribution of members in the first step is obtained by solving an optimization problem in which the constraints are the equilibrium equations and the objective function is the weight. For truss design problems, the approach leads to a linear programming problem. Solution of this linear program allows the areas of the members to be equal to zero so that a new topology with removed members can be found. For the sake of continuity, the present paper will develop the precise formulation of that problem.

For more general optimization problems (nontopology), Guang-Yuan et al.[4] have implemented a displacement-based approach using an optimality criterion for the solution of the displacement field. Again, the following two-step procedure is used:

1) Find the displacements that maximize the strain energy of the truss while satisfying stress and displacement constraints.

2) For the displacements determined in step 1, find the corresponding set of areas that minimize the weight.

All of the areas are assumed to have nonzero lower bounds in Ref. 4, so that the problem is not a topology optimization. For statically determinate trusses, this method produces the true optimum solution in only two steps. However, the two steps have to be repeated sequentially to reach a true optimum in the general case of indeterminate trusses.

One of the earliest implementation of displacement-based optimization was performed by McKeown, who developed a more sophisticated and general application of the method to the design first of composite materials[5] and then of trusses.[6] His work is also a two-level nested approach based on displacement variables at an outer level and on sizing variables at an inner level. The outer-level optimization problem includes the constraints on displacements and strains and can be controlled by any classical optimization algorithm. The inner level is a linear programming problem in terms of the sizing variables and is formulated to satisfy the equilibrium equations while minimizing the weight. McKeown also provides an in-depth description of the nature of the design space. Because this is quite important for the convergence properties of the method, some elements of that description are used in the present paper.

Finally, Striz and Sobieszczanski-Sobieski[7] constructed a more recent approach, demonstrated for beams and trusses, by using surfaces and curves to describe the displacements of their structures. In their approach, coefficients of polynomials that describe the displacements are used as design variables. As was the case in other approaches, the problem is divided into an outer loop (system level), which is used to find the coefficients of the displacement curves, and an inner loop (subsystem level) for the minimization of the weight of each element under stress constraints. To maintain equilibrium, the objective function for the outer loop is chosen to be the minimization of the unbalanced forces at the nodal points of the FE model. The main drawback of the method, based on a FE formulation, is the possibility of having an ill-conditioned system of equations at the system level.

The proposed study is focused on truss structure optimization using the same procedure as was proposed by McKeown.[5,6] An optimum structural design is obtained by searching through a different set of displacement fields at an outer level. For each set, the areas are found in an inner loop such that the weight is minimized. This study will focus on the details of the area and the weight evolution during the optimization process. The effectiveness of gradient-based optimization methods for this kind of problem will be discussed, and the computational efficiency in terms of CPU time will be demonstrated. One of the main emphases of the paper is the application of the method to treat dynamic as well as static response of truss structures.

Description of the Method

The method is based on a nested two-level approach for the solution of displacements and design variables. In the following sections, the two levels, referred to as the inner problem and the outer problem, are discussed in more detailed for both static and dynamic response of a truss structure.

Inner Problem: Optimal Design for a Given Displacement Field

The objective of the inner problem is to find the values of the sizing variables for a given displacement field (compatible with the specified boundary conditions). The sizing variables should be determined such that the equilibrium equations are satisfied while minimizing the total weight. The minimum weight design obtained at this stage is an optimum for the given displacement field, which is modified by the outer problem during the design iterations.

For the design of a truss structure with area design variables A_i, the problem is as follows:

Min

$$W(\boldsymbol{A}) = \sum_{i=1}^{m} \rho l_i A_i$$

subject to governing equilibrium equations, and

$$A_i^L \leq A_i \leq A_i^U, \qquad i = 1, \ldots, m \tag{1}$$

In problem (1), W is the weight, ρ the density, l_i the length of the ith member, and m the number of areas. As mentioned in the Introduction, A_i^L can be equal to zero strictly, which allows topology problems to be solved.

Equilibrium Equations: Static Case

For the truss problem, the nodal equilibrium equations are expressed as a set of linear equation in terms of the cross-sectional areas. For node j, for example, we have

$$\sum_{i=1}^{p} f_{ij,k} + f_{\text{ext},j,k} = 0 \tag{2}$$

where p is the number of elements connected to the node, f_{ij} is the force due to the ith element on node j, and f_{ext} is an external nodal force applied to node j. The subscript k indicates the coordinate axis x, y, or z.

When the basic force stress relation, $f_{ij} = \sigma_{ij} A_i$ is used, Eq. (2) becomes linear in terms of the areas, where the stress σ_{ij} is in the ith member and can be easily computed from the displacements using Hooke's law. The result is a set of $3n$ linear equations for the spatial case and $2n$ equations for plane trusses. These equations are the constraints of the inner optimization problem. Because the weight is also linear in terms of the cross-sectional areas, the problem can be solved using a linear programming algorithm.

Governing Equations: Dynamic Case

The governing equations of undamped vibrations take into account the inertia forces as well as the static forces. Using the FE formulation, we have

$$[\boldsymbol{K} - \omega^2 \boldsymbol{M}]\{\boldsymbol{u}\} = \{\boldsymbol{f}\} \tag{3}$$

where $\boldsymbol{K}$ and $\boldsymbol{M}$ are the stiffness and the mass matrices, respectively; ω is the angular frequency of the excitation; $\boldsymbol{f}$ is the vector of applied forces; and $\boldsymbol{u}$ the displacement vector. Equation (3) is rearranged in a matrix form in terms of the cross-sectional areas. The resulting form is

$$[\boldsymbol{T}] \cdot \{\boldsymbol{A}\} = \{\boldsymbol{f}\} \tag{4}$$

where $\boldsymbol{A}$ is the vector of areas and $\boldsymbol{T}$ is a matrix constructed from the element stiffness and mass matrices and the specified displacements. For example, in the case of a truss structure, all of the components of the ith element stiffness and mass matrices are proportional to the ith cross-sectional area A_i. To illustrate the transformation for system (3) to (4), a simple example is provided. The example consists of a truss with two degrees of freedom (u_1 and u_2) and two areas:

$$\frac{E}{2L}\begin{bmatrix} A_1 + A_2 & A_1 - A_2 \\ A_1 - A_2 & A_1 + A_2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

$$\Leftrightarrow \frac{E}{2L}\begin{bmatrix} u_1 + u_2 & u_1 - u_2 \\ u_1 + u_2 & u_2 - u_1 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{5}$$

where E is the Young's modulus and L the length of both members.

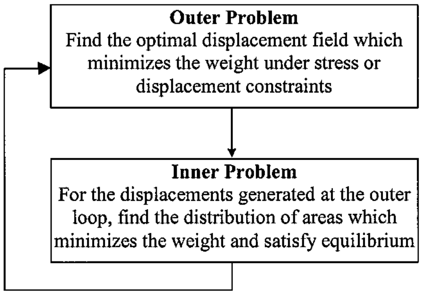

Fig. 1 Flowchart of the optimization process.

The inner system is clearly linear in terms of the cross-sectional areas and yields a set of $3n$ linear equations for three-dimensional trusses and $2n$ equations for planar trusses.

Outer Problem: Search for Optimal Displacement Field

Solution of the inner problem stated in Eq. (1) provides only a temporary optimal solution, which corresponds to the specified displacement field. For a true optimal structure, the displacement field needs to be modified so that lighter and lighter structures are possible. Through those modifications, the displacements are now the new design variables.

Any classical optimization method may be used to determine the optimal displacements. However, as will be discussed shortly, a derivative-based method is preferred because such information is readily available from the solution of the inner problem. The outer-level optimization problem is, therefore, formulated as follows:

$$\min_{u} \quad W(\boldsymbol{u})$$

subject to

$$g_i^L \leq g_i(\boldsymbol{u}) \leq g_i^U, \qquad i = 1, \ldots, m \tag{6}$$

The constraints $g_i(\boldsymbol{u})$ imposed on the problem are the stress or displacement constraints of the global optimization problem with their specified upper and lower bounds. The flow chart of the complete optimization is shown in Fig. 1 and the following points highlight some interesting characteristics of both the inner and outer problems.

Characteristics of the Outer Problem

In the outer problem, the constraints need to be expressed in terms of the displacements. The objective function $W(\boldsymbol{u})$ is an implicit function of the displacement field, which is computed as a result of the inner-problem solution. Note that, whereas the treatment of displacement constraints is obvious, stress constraints, which are typically nonlinear in terms of the sizing variables, are linear in terms of small displacements. This is highly suited for a gradient-based procedure because the stress constraints will be treated exactly. The remaining element is the nature of the total structural weight in terms of the displacements.

Key characteristics of the total structural weight have been described in detail by McKeown.[5,6]

First, $W(\boldsymbol{u})$ is not defined everywhere in the displacement space because there exist some displacement fields that cannot be produced by the structure. Some of the fields are clearly those corresponding to a negative total strain energy, $\boldsymbol{f}^T\boldsymbol{u} < 0$, which is impossible. Part of the design space is then prohibited. In the remaining discussion, the feasible part of the design space will be referred to as the physically feasible domain.

Second, the weight is known to have infinite values close to the boundaries of the physically feasible domain. Therefore, if the initial design is in the physically feasible design space, the optimization search is confined in that domain.

Third, the expression of the dual formulation of the inner problem reveals an interesting feature of the weight function. As part of the solution process, many linear programming packages provide a vector of dual variables $\boldsymbol{\lambda}$. The dimension of the dual variable vector is the same as the displacement vector $\boldsymbol{u}$ and can be thought of as a vector of virtual displacements maximizing the virtual strain energy with restrictions on the strain energy available for each member.

The dual form of the inner problem can be stated as follows:

Max

$$W = f_1\lambda_1 + \cdots + f_m\lambda_m$$

subject to

$$\boldsymbol{B\lambda} \leq \rho \boldsymbol{l} \tag{7}$$

where $\boldsymbol{B}$ is the (m, n) matrix of coefficients of the areas in the inner problem and $\boldsymbol{l}$ is the vector of element lengths of dimension n. It is known that the dual variables give the sensitivity of the objective function with respect to the right-hand-side members of the primal constraint equations.[2] In this case, the right-hand side of the primal problem is the force vector. Therefore, we have

$$\frac{\partial W}{\partial \boldsymbol{f}} = \boldsymbol{\lambda} \tag{8}$$

This result is useful because the differentiation of the weight with respect to the displacement components can be written as follows:

$$\frac{\partial W}{\partial \boldsymbol{u}} = \frac{\partial W}{\partial \boldsymbol{f}}\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\boldsymbol{\lambda} \tag{9}$$

The derivatives of the force with respect to the displacement vector components are obtain by differentiating the equilibrium relation $\boldsymbol{Ku} = \boldsymbol{f}$ with respect to $\boldsymbol{u}$ such that

$$\frac{\partial \boldsymbol{K}}{\partial \boldsymbol{u}}\boldsymbol{u} + \boldsymbol{K} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}} \tag{10}$$

Because the stiffness matrix $\boldsymbol{K}$ depends only on the cross-sectional areas, the first term drops out; therefore,

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}} = \boldsymbol{K} \tag{11}$$

Substituting Eq. (11) into Eq. (9), we finally obtain

$$\frac{\partial W}{\partial \boldsymbol{u}} = \frac{\partial W}{\partial \boldsymbol{f}}\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}} = \boldsymbol{K\lambda} \tag{12}$$

Equation (12) makes the linearization of the weight with respect to the displacement components possible without the need for finite difference derivatives. This provides a sharp reduction in the cost of the optimization because the finite difference formulation would have required solution of the inner problem once for every displacement component perturbation. For structural models with a large number of degrees of freedom, this can lead to a large number of additional linear programming solutions.

Fourth, Eq. (12) does not guarantee the continuity of $W(\boldsymbol{u})$ everywhere in the displacement space. In some case studies, the displacement space will be shown to possess regions corresponding to different distribution of cross-sectional areas. At the boundaries of those regions, the weight will be shown to be nondifferentiable. Nevertheless, the weight exhibits continuity during the whole design history.

Characteristics of the Inner Problem

The inner problem has a linear form in terms of the cross-sectional areas, which can be solved by using the simplex algorithm. In most resizing optimization problems, the areas will have lower and upper bounds. If one area reaches its lower bound and cannot decrease any more, other areas will try to compensate this loss by adjusting their rate of change, causing a change of slope, as shown in Fig. 2, for example, with two areas.

The change in the slope is not considered to be a problem because no constraints depend directly on the evolution of the areas, but on the displacements. It will be shown that these slope changes do not adversely influence (for small-scale problems) the smooth reduction of the weight during the optimization process. It is also shown that, during the evolution of the displacement field, the distribution of areas might change suddenly for a very small change of the displacements. This aspect of the problem is outlined by some numerical examples.

Table 1 Results for the 10-bar truss, static case[a]

CSAs	Displacement based	Classical optimization
1	7.900	7.900
3	8.099	8.100
4	3.900	3.899
7	5.798	5.798
8	5.515	5.515
9	3.676	3.676
10	0.141	0.141
Optimal weight, lb	1497.60	1497.59
Number of iterations	6	7
CPU time, s	0.04	0.15

[a]Members 2, 5, and 6 are at their lower bound of 0.1 in.^{2}.

Table 2 Stress constraints, 25-bar truss, static case

Bar	Compr, psi	Bar	Compr, psi
1	−35092	12, 13	−35092
2–5	−11590	14–17	−6759
6–9	−17305	18–21	−6959
10, 11	−35092	22–25	−11082

Table 3 Load case, 25-bar truss, static case

Node	*x*	*y*	*z*
1	1,000	10,000	−5,000
2	0	10,000	−5,000
3	500	0	0
6	500	0	0

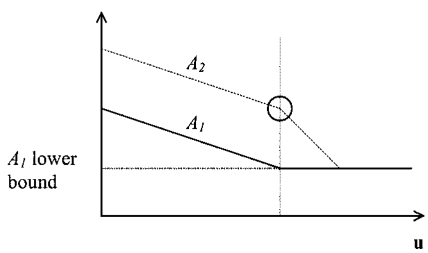

Fig. 2 Change of slope in the area history.

Table 4 Results for the 25-bar truss, static case[a]

CSAs	Displacement based	Classical optimization
2	0.135	0.110
6	3.872	3.877
7	3.198	3.197
8	3.797	3.805
9	3.281	3.283
12	2.187	2.184
13	1.866	1.866
14	0.814	0.806
15	0.690	0.698
16	0.863	0.855
17	0.671	0.678
19	0.086	0.090
21	0.038	0.043
22	3.569	3.566
23	4.089	4.093
24	4.400	4.405
25	3.497	3.491
Optimal weight, lb	449.22	449.17
Number of iterations	30	35
CPU time, s	1.3	4.1

[a]Members 3, 4, 10, 11, 18, 19, and 21 are at their lower bound of 0.01 in.^{2}.

Table 5 Results for the 10-bar truss, dynamic case[a]

CSAs	Displacement based	Classical optimization
1	6.278	6.280
2	5.790	5.789
3	12.144	12.140
7	8.930	8.924
8	0.499	0.510
9	0.140	0.100
10	7.995	7.995
Optimal weight, lb	1775.78	1774.76
Number of iterations	16	16
CPU time, s	0.11	0.61

[a]Members 4–6 and 9 are at their lower bound of 0.1 in.^{2}.

Table 6 Results for the 25-bar truss, dynamic case[a]

CSAs	Displacement based	Classical optimization
2	24.110	24.048
3	0.011	0.011
4	24.111	24.048
5	0.011	0.011
7	0.010	0.010
8	20.022	20.140
9	20.025	20.139
10	5.9675	6.000
11	5.6032	5.634
12	3.004	3.033
13	3.005	3.032
19	10.081	10.059
21	10.081	10.059
22	23.100	23.071
23	23.101	23.071
24	29.197	29.196
25	29.199	29.196
Optimal weight, lb	2951.80	2951.83
Number of iterations	49	66
CPU time, s	2.2	12.1

[a]Members 1, 6, 14–18, and 20 are at their lower bound of 0.01 in.^{2}.

Test Example Results

The described method is demonstrated for 10- and 25-bar truss structures[2] under static and dynamic loads. Both structures are made of aluminum with a Young's modulus of $E = 10^7$ psi and a specific mass of 0.1 lbm/in.^{3}. A lower bound of 0.1 and 0.01 in.^{2} is imposed on the cross-sectional areas of the 10- and 25-bar truss, respectively. Results of the present approach are compared with the results of traditional optimization for both the static and dynamic load cases.

Two different algorithms are used to perform the outer level optimization of the displacement-based optimization (DBM). First, a sequential quadratic programming (SQP) algorithm from the International Mathematical and Statistical Library is used to compare the optimal designs obtained by the DBM and the classical approach and to compare their computational efficiency. The second algorithm used for the outer-level optimization was a sequential linear algorithm (SLP) with small move limits. The purpose of using small move limits is to monitor the evolution of areas and weight for small changes in the displacements during the optimization so that possible discontinuities or nondifferentiable points could be detected. This approach is expected to provide insight as to the relation between the evolution of the areas and the weight in the displacement space. For each example, the optimal design and CPU time are obtained from the SQP solution and presented in Tables 1–6 associated with the examples. The area and weight histories based on the SLP algorithm are also provided for each example. In every case, the final design obtained by using SQP and SLP were the same; hence, only SQP design results are presented.

To demonstrate the use of the DBM to solve topology problems, an example with zero cross-sectional area lower bounds is considered. The example is a 156-bar truss ground structure where elements can be removed to find an optimal topology for given boundary conditions and loading. The minimization of the weight

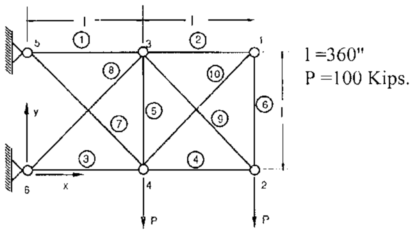

Fig. 3 Truss, 10 bar.

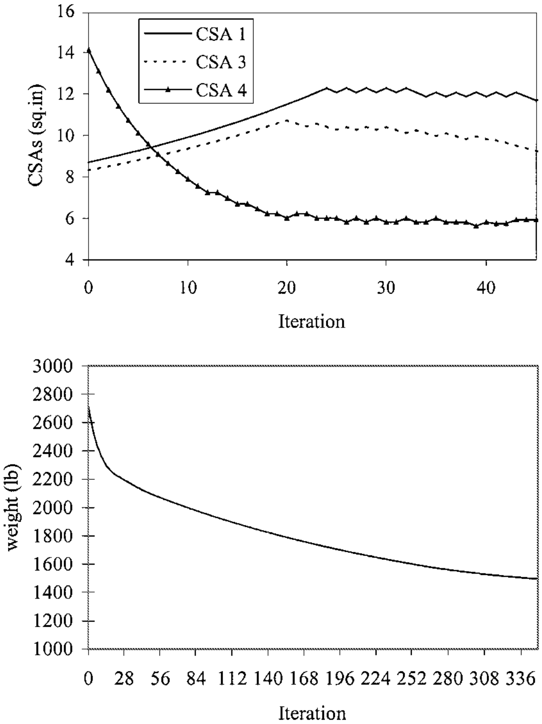

Fig. 4 Evolution of the areas and weight for the 10-bar truss, static case.

subjected to stress constraints is considered. The result is compared to a literature result for the final layout, and the CPU time required is compared to the GENESIS commercial software (Vanderplaats Research and Development, Inc.) CPU time.

Static Load Case

The truss shown in Fig. 3 is subjected to the following stress constraints: -25 ksi $\leq$ stresses in bars 1–10 $\leq$ 25 ksi, and -75 ksi $\leq$ stresses in bar 9 $\leq$ 75 ksi.

Results of both the classical and the DBM approaches are listed in Table 1 for the cross-sectional areas (CSAs) of the members and the optimal weight. A comparison of the computational effort is also provided. The two methods converge to the same solution, but the total CPU time used by the DBM is only a little over 0.04 s compared to almost 0.15 s used by the traditional approach.

The evolution of the 10 areas was monitored during the optimization process based on the SLP algorithm with move limits of 10^{-5} imposed to the displacements. The evolution of three areas is shown in Fig. 4 for the first 70 iterations. One can observe that the CSAs may exhibit some oscillations around an average value. However, this oscillation can be shown to have no effect on the smoothness of the objective function as if all of the variations of areas where self-equilibrating. The history of the minimum weight during each of the displacement cycles is plotted in Fig. 4, which shows a smooth variation of the weight.

The 25-bar truss shown in Fig. 5 is subjected to the stress constraints provided in Table 2. The stress constraint in tension for all of the members is 40 ksi and the applied loads (Table 3) are nodal forces. In addition, ± 0.35-in. displacement constraints have been imposed to nodes 1 and 2 in the x, y, and z directions.

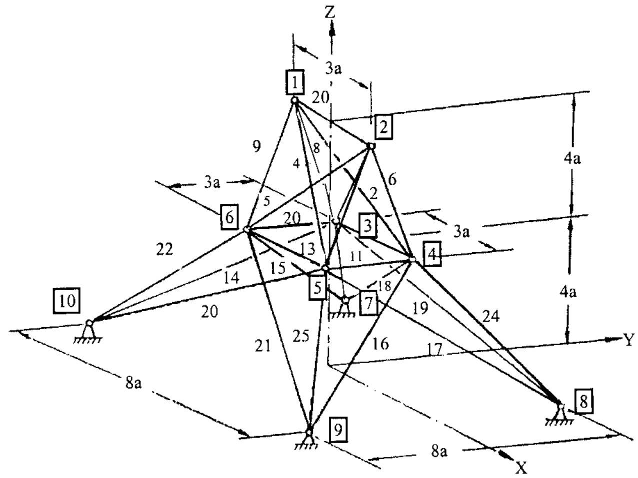

Fig. 5 Truss, 25 bar.

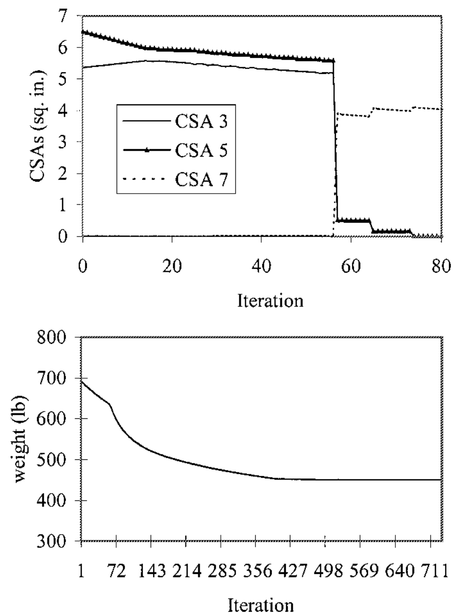

Fig. 6 Evolution of three areas and weight for the 25-bar truss, static case.

The results of both the displacement-based and classical optimization methods are compared in Table 4. There is a good agreement between the methods. History of three CSAs and the weight (based on SLP with move limits of 10^{-5}) is shown in Fig. 6. Again, the CPU time needed for an iteration of the DBM process is smaller than the traditional approach, and the number of cycles for convergence is smaller, leading to a much shorter computational time. The current procedure does not require either the assembly of the stiffness matrix or its inverse, but uses efficient linear programming to solve for the values of the design variables along with the derivatives of the weight with respect to the displacements.

Dynamic Load Case

With an initial value of 2 in.^{2} for the areas, the 10-bar truss is excited at node 1 in the y direction with an angular frequency equal to $\sqrt{10}$ rad/s and subjected to the following displacement and stress constraints: -20 ksi $\leq$ stresses in bars 1–8 $\leq$ 20 ksi, -70 ksi $\leq$ stresses in bar 9 $\leq$ 70 ksi, -20 ksi $\leq$ stresses in bar 10 $\leq$ 20 ksi, -1.0 in. $\leq u_y$ node 1 $\leq$ 1.0 in., and -0.5 in. $\leq u_y$ node 3 $\leq$ 0.5 in.

The results, listed in Table 5, indicate that the CPU time difference between both methods is again significant because no assembly of the mass and stiffness matrix and no inversion of $[\boldsymbol{K} - \omega^2 \boldsymbol{M}]$ are needed. The evolution of the areas shown in Fig. 7 (based on SLP with move limits of 10^{-6}) does not exhibit large discontinuities but has cusps and the same oscillations as encountered in the static case. The weight history (Fig. 7) is again smooth.

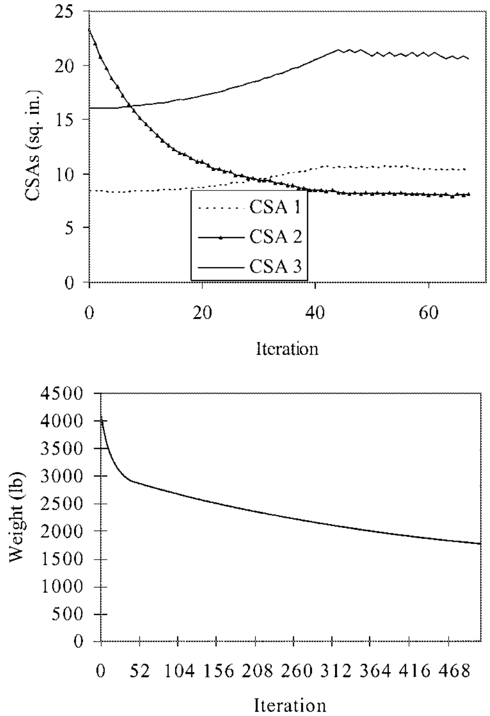

Fig. 7 Evolution of the areas and weight for the 10-bar truss, dynamic case.

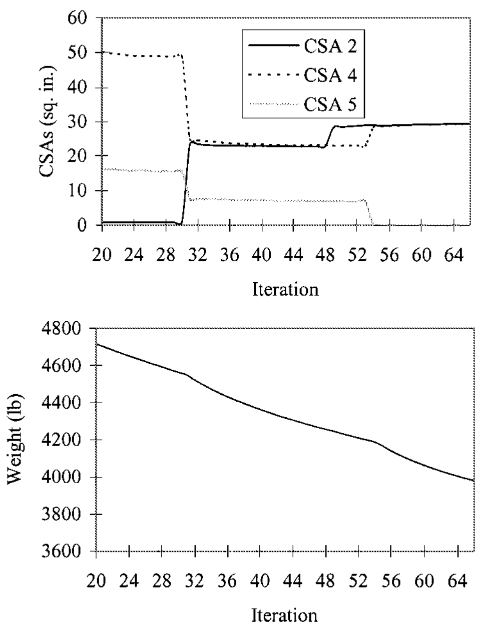

Fig. 8 Evolution of the areas and weight for the 25-bar truss, dynamic case.

The second example is a 25-bar truss excited at node 1 in the x direction by a force of amplitude 10^4 lb with an angular frequency of 10 rad/s. It is subjected to the same stress constraints as in the static case with displacement constraints of ± 0.35 in. on nodes 1, 3, and 6 in the x direction. The results for both optimization methods based on SQP are given in Table 6 and show good agreement. The CPU times are again significantly different; using the DBM is more than five times faster than the traditional approach.

For this example, the history of three members' areas and the weight (Fig. 8) have been monitored for a short range of iterations in which there are some discontinuities, focusing on the cusps appearing on the objective function. Two main discontinuities in

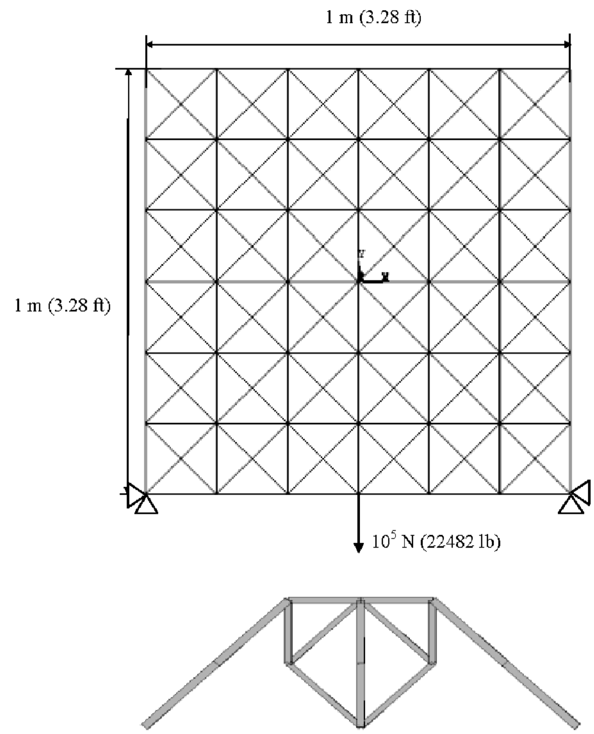

Fig. 9 Truss, 156 bar, and final topology.

slope appear in the weight evolution at iteration 31 and 54, which correspond to two discontinuities in the area evolution. At iteration 48, a discontinuity in the area history has a little effect on the weight evolution. The graphs are based on SLP with move limits of 2×10^{-4}.

Topology Problem

The 156-bar truss and its dimensions are shown in Fig. 9. The two corner points along the bottom edge of the domain are restrained in both the horizontal and vertical directions, and a 10^5-N (22,482-lb) load is applied halfway between the supports along the bottom edge of the domain. The stresses in the members are limited to 200 MPa (29 ksi) in tension and compression.

An initial value of 2×10^{-3} m^2 (3.10 in.^{2}) has been used for all of the areas. An optimal weight of 5.84 kg (12.87 lb) was obtained in 2.2 s and 7 iterations. The optimal layout is shown in Fig. 9. The result is similar to the one obtained by Bendsøe et al.[8] Although the authors were looking for a minimum compliance design, the results are comparable because it can be shown that the minimization of weight subjected to identical stress constraints in tension and compression on all members is equivalent, for the layout, to a minimum compliance design.[9,10] The topology obtained in Ref. 8 has a nondimensional compliance of 2.25, identical to the DBM result.

Because the CPU time required was not provided in Ref. 8, the DBM computational requirements were compared to the commercial GENESIS software CPU time, which is used as a classical optimization tool. GENESIS reached an optimal weight of 5.85 kg (12.89 lb) in 12 iterations, which took 23 s. Here again, the difference in CPU time is important and is due mainly to the presence of FE analysis in the classical approach. The slightly larger weight obtained by GENESIS is due to the formulation of the problem itself in that the areas are reduced to small values but could not be reduced identically to zero to avoid singularities. This is because the version of GENESIS used in the present study did not allow the direct topology optimization problem along with the CSA variation. Currently, the treatment of a topology design in GENESIS requires all of the CSAs to be identical. It is possible to address the type of problem solved by the DBM approach using a two-step process in GENESIS. This approach would consist of finding the minimum compliance topology for a truss with identical section areas for all of the members first. The topology determined by this approach could then be sized to satisfy the weight minimization subjected to stress constraints. Comparison of the CPU time for such a two-step process, which is likely to be even more expensive than the single-step version described earlier, with the current DBM approach would be inappropriate.

Based on the foregoing test cases, it is important to understand more accurately the behavior of the areas and the weight during the optimization by focusing on the topology of the displacement space in the static and dynamic cases. Also, the relation between the feasibility of the displacement field used and the dimensions of the inner problem have to be discussed.

Discussion

CSA History

It is observed that, from one iteration of the outer loop to another, the CSA distribution of the members may experience significant jumps corresponding to slight changes in the set of specified displacements. Existence of such discontinuities does not influence the performance of the optimization because the constraints, which are stresses and displacements, do not depend on the areas. Constraints are only displacement dependent and follow continuously the smooth variations of the displacement variables. As observed in the examples, a discontinuity in a member area is typically associated with discontinuities in other members. The areas remain unchanged for a number of iterations, jump to new values after a while, and remain unchanged until a new distribution of areas is needed. This means that the physically admissible space is divided into several regions corresponding to the different distributions of CSAs. Those domains will be referred to as the inner domains.

Weight History

The structural weight is a strong function of the areas. The sudden jumps observed in the CSAs, from one iteration to another, however, are found to have a small influence on the structural weight. Indeed, only minor discontinuities in the slope of the weight curve are observed. The discontinuities of slope, that is, nondifferentiable points, had no influence, at least for the small-scale problems considered, on the performance of the optimization, because there were only a few of them. What is hidden behind the continuity of the weight is an implicit relationship that exists between optimal weight and displacements. That is, the change in optimal weight can only be small for a small change in displacement. Although this behavior can be intuitively understood when considering the change in the global compliance of the structure, it has to be formally proven.

Comments on the Dynamic Case

The major difference between the dynamic case and the static one is the presence of natural frequencies. Indeed, as the displacements become large near the natural frequency, the design process may oscillate between two successive frequencies. The design space is then made up of several regions whose boundaries are imposed by the natural frequencies. This remark is valid for both the displacement-based method and the classical optimization, but the design space is more complex for the displacement-based method. In fact, for a slight change in the displacement field, the distribution of areas might change drastically (as seen in the static case), causing the distribution of masses and rigidities to change. Then a new set of natural frequencies is obtained, and the design process is prevented from being stuck in a suboptimal solution, as it might be in the classical procedure. The process can then jump from one region to another, but is not user controllable a priori.

Both dynamic examples presented here are somewhat restricted because they use only one excitation frequency rather than a frequency range. This simplifies the problem substantially because, for a well-chosen excitation, the problem of having natural frequencies equal or close to the excitation frequency is avoided. For an undamped system, having the excitation frequency too close to the natural frequency would make the inner problem highly ill conditioned. A more in-depth study is required to establish the influence of the natural frequencies on the design process for cases of forced-vibration analysis.

Inner-Problem Dimensionality

The analysis problem of the classical optimization approach is its inner problem, which is governed by a linear system of equations that are equal in number to the number of unknown displacement components. Solution of such a system is straightforward. In the case of the displacement-based method, on the other hand, transformation of the governing equations so that unknown design variables can be solved for specified displacements may present difficulties. In general, the number of design variables in the problem is different from the number of displacement degrees of freedom, yielding a nonsquare coefficient matrix, the $\boldsymbol{T}$ matrix in Eq. (4), for the transformed inner problem.

For problems in which the number of design variables is larger than the number of displacement degrees of freedom, there will be more unknown design variables than the number of equations, which are the equality constraints of the inner problem. Hence, the inner optimization problem for the minimization of weight as described earlier is possible. This is the case for most truss structures because two or more members are connected to the same node. Indeed, for the 10- and 25-bar trusses considered, the number of design variables is higher than the number of degrees of freedom.

On the other hand, in the case of problems in which the number of design variables is less than the number of displacement degrees of freedom, we have more equality constraints than the number of unknown design variables in the inner problem. Such a system is overconstrained and, in general, there is no solution for the unknown design variables for an arbitrary displacement field. To overcome this difficulty, it may be possible to relax the equality constraints and not require the satisfaction of the equilibrium until the final optimal structure is achieved.

Conclusions

A generalized two-level DBM method and its application to truss structures have been presented. For all examples treated so far, the results of the method show a good agreement with the results of the classical optimization procedure. Observations of the characteristics of the CSAs and the weight variations during the optimization process are made. Despite significant discontinuities in the CSAs, it has been shown that the objective function, stress, and displacement constraints are continuous functions of the displacement variables. However, these discontinuities might generate nondifferentiable points during the optimization process, and their influence on larger scale problems has to be investigated.

The use of dual variables obtained as part of the inner optimization problem proved to be useful in computing the exact gradient of the objective function with respect to the displacement, avoiding the use of finite differences. For the static, the dynamic, and the topology cases, substantial reductions in CPU time were obtained. A more in-depth study is needed for the dynamic case in which the natural frequencies make the design space more difficult to navigate.

Currently, research continues on the extension of the method to problems integrating geometric and material nonlinearities. The procedure, which specifies the displacements, is particularly well suited for that class of problems.

Acknowledgments

This research was partially funded by the Design Optimization and Vehicle Safety Research Department of the Ford Motor Company under the University Research Program. The Project Monitor was Ren-Jye Yang. The authors also wish to acknowledge Juan Pablo Leiva from Vanderplaats Research and Development, Inc., for his help in running the GENESIS solutions.

References

[1]Haftka, R. T., and Kamat, M. P., "Simultaneous Nonlinear Structural Analysis and Design," *Computational Mechanics*, Vol. 4, No. 6, 1989, pp. 409–416.

[2]Haftka, R. T., and Gürdal, Z., *Elements of Structural Optimization*, Kluwer Academic, Boston, 1992, Chap. 3.

[3]Ringertz, U. T., "On Topology Optimization of Trusses," *Engineering Optimization*, Vol. 9, No. 3, 1985, pp. 209–218.

[4]Guang-Yuan, W., Zheng-Yuan, Z., and Huo Da, H., "A Two-Phase Optimization Method for Minimum-Weight Design of Trusses," *Engineering Optimization*, Vol. 8, No. 1, 1984, pp. 55–67.

[5]McKeown, J. J., "Optimal Composite Structures by Deflection Variable Programming," *Computer Methods in Applied Mechanics and Engineering*, Vol. 12, No. 3, 1977, pp. 155–179.

[6]McKeown, J. J., "The Design of Optimal Trusses via Sequences of Optimal Fixed Displacement Structures," *Engineering Optimization*, Vol. 14, 1989, pp. 159–178.

[7]Striz, A. G., and Sobieszczanski-Sobieski, J., "Displacement Based Multilevel Structural Optimization," AIAA Paper 96-4098, Sept. 1996.

[8]Bendsøe, M. P., Ben-Tal, A., and Haftka, R. T., "New Displacement-Based Methods for Optimal Truss Topology Design," *Proceedings of the 32nd AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Material Conference*, AIAA, Washington, DC, 1991, pp. 684–696.

[9]Sankaranarayanan, S., Haftka, R. T., and Kapania, R. K., "Truss Topology Optimization with Stress and Displacement Constraints," *AIAA Journal*, Vol. 32, No. 2, 1994, pp. 420–424.

[10]Sankaranarayanan, S., "Topology Optimization with Simultaneous Analysis and Design," Ph.D. Dissertation, Virginia Polytechnic Inst. and State Univ., Blacksburg, VA, Aug. 1992.

A. Messac
Associate Editor